NONAXIAL TEMPERATURE DISTRIBUTION IN AN INFINITELY LONG ORTHOTROPIC CYLINDER

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1. Consider an infinitely long orthotropic homogeneous hollow cylinder of outside radius R + h/2 and internal radius R − h/2 in a cylindrical coordinate system z, β, r (the z-axis lies along the axis of the cylinder, while β and r are polar coordinates).

Initially (t = 0), the cylinder has a constant temperature T_0, while the boundary conditions at the surface are independent of z and are given in the form [1]

$$d_1\frac{\partial T}{\partial r}-g_1'T=F_1' \quad \text{at} \quad r=R-{}^1/_2h\,,$$
$$d_2\frac{\partial T}{\partial r}+g_2'T=F_2' \quad \text{at} \quad r=R+{}^1/_2h\,,$$

where $F_j'=F_j'(\beta)$ are given functions of β, while d_j and g_j' (j = 1, 2) are positive constant coefficients (we rule out the cases in which d_1 and g_1' or d_2 and g_2' are zero simultaneously).

The material is assumed to be orthotropic in thermophysical properties, with the principal axes of the thermal conductivity coincident with the principal geometrical directions.

The heat-conduction equation then takes the form

$$\frac{\lambda_1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\lambda_2}{r^2}\frac{\partial^2T}{\partial\beta^2}=\rho c_0\frac{\partial T}{\partial t},$$

where $\lambda_1>0$ and $\lambda_2>0$ are the thermal conductivities along the r and β axes, respectively, while ρ is density and c_0 is specific heat.

We convert to the dimensionless temperature $\vartheta\equiv T/T^\circ$, in which $T^\circ=T_0$ if $T_0\neq 0$, while any fixed temperature $T^\circ>0$ may be taken if this is not so, and also to the dimensionless coordinate $y\equiv r/R$ and to the dimensionless time $\tau\equiv t/t_0$, in which t_0 = constant > 0. We also introduce the symbols

$$\frac{h}{R}\equiv 2\delta,\quad \frac{R^2\rho c_0}{\lambda_1 t_0}\equiv m^2,\quad \frac{T_0}{T^\circ}\equiv\vartheta^\circ,\quad \beta\Lambda_{12}\equiv\varphi,$$

$$RF_j'(\beta)/T^\circ\equiv F_j(\varphi),\quad Rg_j'\equiv g_j\quad (j=1,2),\quad \begin{array}{l}\sqrt{\lambda_1/\lambda_2}=\Lambda_{12},\\ \sqrt{\lambda_2/\lambda_1}=\Lambda_{21}.\end{array}$$

The problem then reduces to integration of

$$\frac{\partial^2\vartheta}{\partial y^2}+\frac{1}{y}\frac{\partial\vartheta}{\partial y}+\frac{1}{y^2}\frac{\partial^2\vartheta}{\partial\varphi^2}-m^2\frac{\partial\vartheta}{\partial\tau}=0 \tag{1.1}$$

subject to the initial condition

$$\vartheta=\vartheta^\circ \quad \text{at} \quad \tau=0, \tag{1.2}$$

and to the boundary conditions

$$d_1\frac{\partial\vartheta}{\partial y}-g_1\vartheta=F_1 \quad \text{at} \quad y=y_1=1-\delta,$$
$$d_2\frac{\partial\vartheta}{\partial y}+g_2\vartheta=F_2 \quad \text{at} \quad y=y_2=1+\delta.$$

In all cases of practical interest, $F_1(\varphi)$ and $F_2(\varphi)$ satisfy the Dirichlet conditions and can be expanded as Fourier series in the range $0\le\varphi\le 2\pi\Lambda_{12}$:

$$F_j(\varphi)=\frac{a_0^{(j)}}{2}+\sum_{k=1}^{\infty}a_k^{(j)}\cos(k\Lambda_{21}\varphi)+$$
$$+\sum_{k=1}^{\infty}b_k^{(j)}\sin(k\Lambda_{21}\varphi)\quad (j=1,2),$$

$$a_k^{(j)}=\frac{\Lambda_{21}}{\pi}\int_0^{2\pi\Lambda_{12}}F_j(\varphi)\cos(k\Lambda_{21}\varphi)\,d\varphi,$$

$$b_k^{(j)}=\frac{\Lambda_{21}}{\pi}\int_0^{2\pi\Lambda_{12}}F_j(\varphi)\sin(k\Lambda_{21}\varphi)\,d\varphi\quad (j=1,2). \tag{1.3}$$

We also represent $\vartheta(y,\varphi,\tau)$ as a trigonometric series

$$\vartheta(y,\varphi,\tau)=\frac{\vartheta_0(y,\tau)}{2}+\sum_{k=1}^{\infty}\vartheta_k(y,\tau)\cos(k\Lambda_{21}\varphi)+$$
$$+\sum_{k=1}^{\infty}\vartheta_k^*(y,\tau)\sin(k\Lambda_{21}\varphi). \tag{1.4}$$

The object of this paper is to determine the coefficients.

2. The Laplace transform of a function in the plane of the complex variable p is

$$\Theta(p)=\int_0^\infty e^{-p\tau}\vartheta(\tau)\,d\tau,\qquad \mathrm{Re}(p)>0.$$

Converting to the transforms in the basic relations and using (1.2), we have:

a) for the equation of heat conduction

$$\frac{\partial^2\Theta}{\partial y^2}+\frac{1}{y}\frac{\partial\Theta}{\partial y}+\frac{1}{y^2}\frac{\partial^2\Theta}{\partial\varphi^2}-m^2p\Theta=-m^2\vartheta^\circ; \tag{2.1}$$

b) for the boundary conditions

$$d_1\frac{\partial\Theta}{\partial y}-g_1\Theta=\frac{F_1}{p}\qquad \text{for}\quad y=y_1=1-\delta,$$
$$d_2\frac{\partial\Theta}{\partial y}+g_2\Theta=\frac{F_2}{p}\qquad \text{for}\quad y=y_2=1+\delta; \tag{2.2}$$

c) for the series expansion of the transform of the desired function

$$\Theta(y,\varphi)=\frac{\Theta_0(y)}{2}+\sum_{k=1}^{\infty}\Theta_k(y)\cos(k\Lambda_{21}\varphi)+$$
$$+\sum_{k=1}^{\infty}\Theta_k^*(y)\sin(k\Lambda_{21}\varphi). \tag{2.3}$$

We substitute (1.3) and (2.3) into (2.1) and (2.2) to get

$$\frac{d^2\Theta_0}{dy^2}+\frac{1}{y}\frac{d\Theta_0}{dy}-m^2p\Theta_0=-2m^2\vartheta^\circ, \tag{2.4}$$

with the boundary conditions

$$d_1\frac{d\Theta_0}{dy}-g_1\Theta_0=\frac{a_0^{(1)}}{p}\qquad \text{for}\quad y=y_1=1-\delta,$$
$$d_2\frac{d\Theta_0}{dy}+g_2\Theta_0=\frac{a_0^{(2)}}{p}\qquad \text{for}\quad y=y_2=1+\delta \tag{2.5}$$

for Θ_0, and also the homogeneous differential equation

$$\frac{d^2(\;)}{dy^2}+\frac{1}{y}\frac{d(\;)}{dy}-\left(m^2p+\frac{\nu^2}{y^2}\right)(\;)=0, \tag{2.6}$$

with the boundary conditions

$$d_1\frac{d\Theta_k}{dy}-g_1\Theta_k=\frac{a_k^{(1)}}{p},\qquad d_1\frac{d\Theta_k^*}{dy}-g_1\Theta_k^*=\frac{b_k^{(1)}}{p}$$

for $y=y_1=1-\delta$,

$$d_2 \frac{d\Theta_k}{dy} + g_2\Theta_k = \frac{a_k^{(2)}}{p}, \quad d_2\frac{d\Theta_k^*}{dy} + g_2\Theta_k^* = \frac{b_k^{(2)}}{p}$$

$$\text{for } y = y_2 = 1 + \delta \tag{2.7}$$

for Θ_k and Θ_k^* ($k = 1, 2, \ldots$).

In (2.6) we have used the symbol $\nu^2 \equiv k^2 \lambda_2/\lambda_1 > 0$, in which ν, in general, may be any real number (integer or fraction).

The solutions to (2.4) and (2.6) are represented by Bessel functions of orders zero and ν, respectively.

3. The general solution to (2.4) can be put as [2]

$$\Theta_0 = A_0J_0(im\sqrt{p}y) + B_0N_0(im\sqrt{p}y) + 2\vartheta^\circ / p$$

in which $J_0(\ )$ and $N_0(\ )$ are, respectively, Bessel and Neumann functions of zero order [3].

We deduce A_0 and B_0 from (2.5) and set

$$imy_1\sqrt{p} \equiv \xi, \quad imy_2\sqrt{p} = \varepsilon\xi, \quad imy\sqrt{p} = \chi$$

$$\left(\frac{y_2}{y_1} = \frac{1+\delta}{1-\delta} \equiv \varepsilon > 1, \ \frac{\chi}{\xi} \equiv \frac{y}{y_1}\right) \tag{3.1}$$

to get

$$\Theta_0(p) = \frac{2\vartheta^\circ}{p} + \frac{\Omega_0(\xi)}{p\Delta_0(\xi)},$$

$$\Omega_0(\xi)/y_1 = \{(a_0^{(1)} + 2g_1\vartheta^\circ)[g_2y_1N_0(\varepsilon\xi) - d_2\xi N_1(\varepsilon\xi)] +$$
$$+ (a_0^{2} - 2g_2\vartheta^\circ)[g_1y_1N_0(\xi) + d_1\xi N_1(\xi)]\}J_0(\chi) -$$
$$- \{(a_0^{(2)} - 2g_2\vartheta^\circ)[g_1y_1J_0(\xi) + d_1\xi J_1(\xi)] +$$
$$+ (a_0^{(1)} + 2g_1\vartheta^\circ)[g_2y_1J_0(\varepsilon\xi) - d_2\xi J_1(\varepsilon\xi)]\}N_0(\chi),$$
$$\Delta_0(\xi) = [g_1y_1J_0(\xi) + d_1\xi J_1(\xi)] \times$$
$$\times [-g_2y_1N_0(\varepsilon\xi) + d_2\xi N_1(\varepsilon\xi)] + [g_2y_1J_0(\varepsilon\xi) -$$
$$- d_2\xi J_1(\varepsilon\xi)][g_1y_1N_0(\xi) + d_1\xi N_1(\xi)]. \tag{3.2}$$

To Θ_0 we apply the inverse Laplace transformation

$$\vartheta_0(\tau) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\tau p}\,\Theta_0(p)\,dp \tag{3.3}$$

to find ϑ_0, i.e., one of the desired functions of the series of (1.4).

The integrand in (3.3) is a single-valued function of p with a pole at $p = 0$ and with simple poles at $p = p_{0n} \equiv -\xi^2_{0n}/m^2y_1^2$, where $\pm\xi_{0n}$ denotes the roots (all real and simple [1]) of

$$\Delta_0(\xi) = 0. \tag{3.4}$$

If

$$g_1d_2 + g_2\varepsilon d_1 + g_1g_2\varepsilon y_1 \ln\varepsilon \neq 0, \tag{3.5}$$

then the point $p = 0$ will be a pole of first order. We substitute (3.2) into (3.3) and apply the standard formula to calculate the residues [2] so that we finally obtain

$$\vartheta_0(y, \tau) = 2\left\{\vartheta^\circ + f_0 + \sum_{n=1}^{\infty}[C_{0n}J_0(\xi_{0n}y/y_1) + \right.$$
$$\left. + D_{0n}N_0(\xi_{0n}y/y_1)]\exp\left(-\frac{\xi_{0n}^2\tau}{m^2y_1^2}\right)\right\},$$

$$f_0 = \Big[\varepsilon(a_0^{(2)} - 2g_2\vartheta^\circ)[d_1 + g_1y_1(\ln y - \ln y_1)] -$$
$$- (a_0^{(1)} + 2g_1\vartheta^\circ)[d_2 - g_2\varepsilon y_1(\ln y - \ln\varepsilon - \ln y_1)]\Big] \times$$
$$\times \Big[2(g_1d_2 + g_2d_1\varepsilon + g_1g_2\varepsilon y_1\ln\varepsilon)\Big]^{-1},$$

$$C_{0n} = \frac{y_1}{\Delta_{0n}'}\{(a_0^{(1)} + 2g_1\vartheta^\circ)[g_2y_1N_0(\varepsilon\xi_{0n}) - d_2\xi_{0n}N_1(\varepsilon\xi_{0n})] +$$
$$+ (a_0^{(2)} - 2g_2\vartheta^\circ)[g_1y_1N_0(\xi_{0n}) + d_1\xi_{0n}N_1(\xi_{0n})]\},$$

$$D_{0n} = -\frac{y}{\Delta_{0n}'}\{(a_0^{(2)} - 2g_2\vartheta^\circ)[g_1y_1J_0(\xi_{0n}) + d_1\xi_{0n}J_1(\xi_{0n})] +$$
$$+ (a_0^{(1)} + 2g_1\vartheta^\circ)[g_2y_1J_0(\varepsilon\xi_{0n}) - d_2\xi_{0n}J_1(\varepsilon\xi_{0n})]\},$$

$$\Delta_{0n}' = 2p\frac{d\Delta_0(\xi)}{dp}\Big|_{p=p_{0n}} =$$
$$= \xi_{0n}\{[-d_1\xi_{0n}J_0(\xi_{0n}) + g_1y_1J_1(\xi_{0n})][g_2y_1N_0(\varepsilon\xi_{0n}) -$$
$$- d_2\xi_{0n}N_1(\varepsilon\xi_{0n})] + [-d_1\xi_{0n}N_0(\xi_{0n}) + g_1y_1N_1(\xi_{0n})] \times$$
$$\times [-g_2y_1J_0(\varepsilon\xi_{0n}) + d_2\xi_{0n}J_1(\varepsilon\xi_{0n})] - \varepsilon[d_2\xi_{0n}J_0(\varepsilon\xi_{0n}) +$$
$$+ g_2y_1J_1(\varepsilon\xi_{0n})][g_1y_1N_0(\xi_{0n}) + d_1\xi_{0n}N_1(\xi_{0n})] +$$
$$+ \varepsilon[d_2\xi_{0n}N_0(\varepsilon\xi_{0n}) + g_2y_1N_1(\varepsilon\xi_{0n})] \times$$
$$\times [g_1y_1J_0(\xi_{0n}) + d_1\xi_{0n}J_1(\xi_{0n})]\}. \tag{3.6}$$

Here and subsequently we use the limiting relations

$$\lim_{x\to 0}\frac{J_\nu(\mu x)}{x^\omega J_n(x)} = \begin{cases} 0 & \text{for } \nu - n - \omega > 0 \\ 2^{n-\nu}\mu^\nu\,\Gamma(n+1)/\Gamma(\nu+1) & \text{for } \nu - n - \omega = 0 \\ \infty & \text{for } \nu - n - \omega < 0 \end{cases}$$

$$\lim_{x\to 0}\frac{x^\omega N_n(x)}{N_\nu(\mu x)} = \begin{cases} 0 & \text{for } \nu - n + \omega > 0 \\ 2^{n-\nu}\mu^\nu\,\Gamma(n)/\Gamma(\nu) & \text{for } \nu - n + \omega = 0 \\ \infty & \text{for } \nu - n + \omega < 0 \end{cases}$$

which are true for integer and fractional values of $\nu \geq 0$, $n \geq 0$, $\omega \geq 0$.

If (3.5) is not obeyed (which is possible only if g_1 and g_2 are zero simultaneously when the restrictions of the previous section on d_j and g_j' are applied), the point $p = 0$ is a second-order pole. Then in (3.6) we should assume that

$$f_0 = \frac{a_0^{(2)}d_1\varepsilon - a_0^{(1)}d_2}{d_1d_2m^2y_1(\varepsilon^2 - 1)}\tau + \frac{a_0^{(1)}y_1}{4d_1(\varepsilon^2 - 1)} \times$$
$$\times \left[1 + 2\varepsilon^2\left(1 - \ln y_1 + \ln y - \frac{\varepsilon^2}{\varepsilon^2 - 1}\ln\varepsilon\right)\right] +$$
$$+ \frac{a_0^{(2)}\varepsilon y_1}{4d_2(\varepsilon^2 - 1)}\left[\varepsilon^2 + 2\left(1 + \ln y_1 - \ln y + \frac{\varepsilon^2}{\varepsilon^2 - 1}\ln\varepsilon\right)\right]. \tag{3.7}$$

The linear function of time in (3.7) may lead to an unbounded increase in the absolute value of the temperature of the hollow cylinder. This case can occur if there is no heat transfer at either boundary ($g_1 = g_2 = 0$) or if the amount of heat introduced via one bounding surface is not equal to the amount lost through the other ($a_0^{(2)}d_1\varepsilon \neq a_0^{(1)}d_2$).

4. As Θ_k and Θ_k^* ($k = 1, 2, \ldots$) are defined by (2.6) and satisfy (2.7), while differing only in the values of the coefficients $a_k^{(j)}$ and $b_k^{(j)}$, we consider only $\vartheta_k(y, \tau)$. In the final expression for this, we replace the $a_k^{(j)}$ and $b_k^{(j)}$ and thus get the expression for $\vartheta_k^*(y, \tau)$.

The features of the Bessel functions due to the integer or fractional nature of ν [4] allow us to put the solution to (2.6) for the present case in terms of the symbols of (3.1) as

$$\Theta_k = A_kJ_\nu(\chi) + B_kN_\nu(\chi), \tag{4.1}$$

in which $J_\nu(\ )$ and $N_\nu(\ )$ are Bessel and Neumann functions [3] of order ν.

If $\nu + 1$ is a natural number, we have either an isotropic cylinder ($\nu = k$) or the particular case of orthotropy where $\Lambda_{21}^2 = \lambda_2/\lambda_1$ is the square of a natural number.

We deduce A_k and B_k from (2.7) and substitute these into (4.1) to get

$$\Theta_k = \frac{\Omega_k(\xi)}{p\Delta_k(\xi)},$$

$$\Omega_k(\xi)/y_1 = \{a_k^{(1)}[(d_2\nu + g_2\varepsilon y_1)N_\nu(\varepsilon\xi) - \varepsilon\xi d_2 N_{\nu+1}(\varepsilon\xi)] -$$

$$- a_k^{(2)}\varepsilon[(d_1\nu - g_1y_1)N_\nu(\xi) - d_1\xi N_{\nu+1}(\xi)]\}J_\nu(\chi) +$$

$$+ \{a_k^{(2)}\varepsilon[(d_1\nu - g_1y_1)J_\nu(\xi) - \xi d_1 J_{\nu+1}(\xi)] -$$

$$- a_k^{(1)}[(d_2\nu + g_2\varepsilon y_1)J_\nu(\varepsilon\xi) - \varepsilon\xi d_2 J_{\nu+1}(\varepsilon\xi)]\}N_\nu(\chi),$$

$$\Delta_k(\xi) = [(d_1\nu - g_1y_1)J_\nu(\xi) - \xi d_1 J_{\nu+1}(\xi)] \times$$

$$\times [(d_2\nu + g_2\varepsilon y_1)N_\nu(\varepsilon\xi) - \varepsilon\xi d_2 N_{\nu+1}(\varepsilon\xi)] -$$

$$- [(d_1\nu - g_1y_1)N_\nu(\xi) - \xi d_1 N_{\nu+1}(\xi)] \times$$

$$\times [(d_2\nu + g_2\varepsilon y_1)J_\nu(\varepsilon\xi) - \varepsilon\xi d_2 J_{\nu+1}(\varepsilon\xi)]. \quad (4.2)$$

Inverse transformation from Θ_k gives us $\vartheta_k(y, \tau)$.

In this case, the integrand is a single-valued function of p with poles at $p = 0$ and $p = p_{kn} \equiv \xi_{kn}^2/m^2y_1^2$, in which $\pm\xi_{kn}$ denotes the roots of $\Delta_k(\xi) = 0$. If

$$\frac{\nu y_1(\varepsilon d_1 g_2 + d_2 g_1)}{d_1 d_2 \nu^2 + g_1 g_2 \varepsilon y_1^2} \neq \frac{1 - \varepsilon^{2\nu}}{1 + \varepsilon^{2\nu}}, \quad (4.3)$$

the pole at $p = 0$ will be simple, since

$$\lim_{p\to 0}(p\Theta_k) = f_k, \quad f_k = y_1 \frac{l_1 + l_2}{l_3} \neq \infty,$$

$$l_1 = [a_k^{(1)}\varepsilon^{-\nu}(g_2\varepsilon y_1 - \nu d_2) + a_k^{(2)}\varepsilon(g_1y_1\nu d_1)](y/y_1)^\nu,$$

$$l_2 = [a_k^{(2)}\varepsilon(\nu d_1 - g_1y_1) - a_k^{(1)}\varepsilon^\nu(\nu d_2 + g_2\varepsilon y_1)](y_1/y)^\nu,$$

$$l_3 = (\nu d_1 - g_1y_1)(g_2\varepsilon y_1 - \nu d_2)\varepsilon^{-\nu} + (g_1y_1 + \nu d_1)(g_2\varepsilon y_1 + \nu d_2)\varepsilon^\nu.$$

In this case, (4.3) is always obeyed, since $\varepsilon > 1$ and $\nu > 0$, while the conditions $d_j > 0$ and $g_j > 0$ have been given in section 1. Note that condition (3.5) may be obtained by passing to the limit in (4.3).

We substitute (4.2) into the inverse transformation and use the subtraction theorem to get, for the case of simple roots ξ_{kn},

$$\vartheta_k(y, \tau) = f_k + \sum_{n=1}^{\infty}[C_{kn}J_\nu(\xi_{kn}y/y_1) +$$

$$+ D_{kn}N_\nu(\xi_{kn}y/y_1)]\exp\left(\frac{-\xi_{kn}^2\tau}{m^2y_1^2}\right),$$

$$\vartheta_k(y, \tau) = f_k^* + \sum_{n=1}^{\infty}[C_{kn}^*J_\nu(\xi_{kn}y/y_1) +$$

$$+ D_{kn}^*N_\nu(\xi_{kn}y/y_1)]\exp\left(\frac{-\xi_{kn}^2\tau}{m^2y_1^2}\right), \quad (4.4)$$

where the constants C_{kn} and D_{kn} ($k = 1, 2, \ldots$) are defined by

$$C_{kn} = \frac{2y_1}{\Delta'_{kn}}\{a_k^{(1)}[-d_2\varepsilon\xi_{kn}N_{\nu+1}(\varepsilon\xi_{kn}) +$$

$$+ (d_2\nu + g_2\varepsilon y_1)N_\nu(\varepsilon\xi_{kn})] + a_k^{(2)}\varepsilon[d_1\xi_{kn}N_{\nu+1}(\xi_{kn}) +$$

$$+ (-d_1\nu + g_1y_1)N_\nu(\xi_{kn})]\},$$

$$D_{kn} = \frac{2y_1}{\Delta'_{kn}}\{a_k^{(2)}\varepsilon[-d_1\xi_{kn}J_{\nu+1}(\xi_{kn}) +$$

$$+ (d_1\nu - g_1y_1)J_\nu(\xi_{kn})] + a_k^{(1)}[d_2\varepsilon\xi_{kn}J_{\nu+1}(\varepsilon\xi_{kn}) -$$

$$- (d_2\nu + g_2\varepsilon y_1)J_\nu(\varepsilon\xi_{kn})]\},$$

$$\Delta'_{kn} = 2p\frac{d[\Delta_k(\xi)]}{dp}\bigg|_{p=p_{kn}} =$$

$$= [(d_1\nu^2 - g_1y_1\nu - d_1\xi_{kn}^2)J_\nu(\xi_{kn}) + g_1y_1\xi_{kn}J_{\nu+1}(\xi_{kn})] \times$$

$$\times [(d_2\nu + g_2\varepsilon y_1)N_\nu(\varepsilon\xi_{kn}) - d_2\varepsilon\xi_{kn}N_{\nu+1}(\varepsilon\xi_{kn})] +$$

$$+ [(d_1\nu^2 - g_1y_1\nu - d_1\xi_{kn}^2)N_\nu(\xi_{kn}) + g_1y_1\xi_{kn}N_{\nu+1}(\xi_{kn})] \times$$

$$\times [-(d_2\nu + g_2\varepsilon y_1)J_\nu(\varepsilon\xi_{kn}) + d_2\varepsilon\xi_{kn}J_{\nu+1}(\varepsilon\xi_{kn})] +$$

$$+ [(d_2\nu^2 + g_2\varepsilon y_1\nu - d_2\varepsilon^2\xi_{kn}^2)J_\nu(\varepsilon\xi_{kn}) - g_2y_1\varepsilon^2\xi_{kn}J_{\nu+1}(\varepsilon\xi_{kn})] \times$$

$$\times [(-d_1\nu + g_1y_1)N_\nu(\xi_{kn}) + d_1\xi_{kn}N_{\nu+1}(\xi_{kn})] +$$

$$+ [(d_2\nu^2 + g_2\varepsilon y_1\nu - d_2\varepsilon^2\xi_{kn}^2)N_\nu(\varepsilon\xi_{kn}) - g_2y_1\varepsilon^2\xi_{kn}N_{\nu+1}(\varepsilon\xi_{kn})] \times$$

$$\times [(d_1\nu - g_1y_1)J_\nu(\xi_{kn}) - d_1\xi_{kn}J_{\nu+1}(\xi_{kn})],$$

while f_k^*, C_{kn}^*, D_{kn}^* are derived, respectively, from f_k, C_{kn}, D_{kn} by replacing the $a_k^{(j)}$ by $b_k^{(j)}$.

We substitute (3.6) and (4.4) into (1.4) to get $\vartheta(y, \varphi, \tau)$ as

$$\vartheta = \vartheta^\circ + \sum_{k=0}^{\infty}\left\{f_k + \sum_{n=1}^{\infty}[C_{kn}J_\nu(\xi_{kn}y/y_1) +\right.$$

$$\left. + D_{kn}N_\nu(\xi_{kn}y/y_1)]\exp\frac{-\xi_{kn}^2\tau}{m^2y_1^2}\right\}\cos(\nu\varphi) +$$

$$+ \sum_{k=1}^{\infty}\left\{f_k^* + \sum_{n=1}^{\infty}[C_{kn}^*J_\nu(\xi_{kn}y/y_1) +\right.$$

$$\left. + D_{kn}^*N_\nu(\xi_{kn}y/y_1)]\exp\frac{-\xi_{kn}^2\tau}{m^2y_1^2}\right\}\sin(\nu\varphi). \quad (4.5)$$

From (4.5) we readily get the steady-state temperature distribution as

$$\vartheta(y, \varphi, \infty) = \vartheta^\circ + \sum_{k=0}^{\infty} f_k\cos(\nu\varphi) + \sum_{k=1}^{\infty} f_k^*\sin(\nu\varphi).$$

From (4.5) we also get as a special case the solution for axially symmetric boundary conditions at the surfaces $y = y_1$ and $y = y_2$. Here it is sufficient to take the term $k = 0$ in (4.5). Then we have

$$\vartheta(y, \tau) = \vartheta^\circ + f_0 + \sum_{n=1}^{\infty}[C_{0n}J_0(\xi_{0n}y/y_1) +$$

$$+ D_{0n}N_0(\xi_{0n}y/y_1)]\exp\frac{-\xi_{0n}^2\tau}{m^2y_1^2},$$

which agrees with the known solution [1].

5. As solution (4.5) to (1.1) is obtained formally by term-by-term differentiation (twice with respect to y and twice with respect to φ) of the series in (1.4), we need to demonstrate uniform convergence of the following series in the region $y_2 \geq y \geq y_1$, $2\pi \geq \varphi \geq 0$, $\tau > 0$:

$$S_1 \equiv \sum_{k=0}^{\infty}\frac{\partial^2 u_k}{\partial y^2}, \quad S_2 \equiv \sum_{k=0}^{\infty}\frac{\partial^2 u_k}{\partial\varphi^2},$$

$$S_3 \equiv \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{\partial^2 v_{kn}}{\partial y^2}, \quad S_4 \equiv \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{\partial^2 v_{kn}}{\partial\varphi^2}, \quad (5.1)$$

$$u_k \equiv f_k\cos(\nu\varphi), \quad (5.2)$$

$$v_{kn} \equiv [C_{kn}J_\nu(\xi_{kn}y/y_1) + D_{kn}(\xi_{kn}y/y_1)] \times$$

$$\times \exp\frac{-\xi_{kn}^2\tau}{m^2y_1^2}\cos(\nu\varphi). \quad (5.3)$$

From (5.2) we get

$$\sum_{k=k_0}^{\infty}\frac{\partial^2 u_k}{\partial y^2} = \sum_{k=k_0}^{\infty}\nu(a_k^{(1)}M_k^{(1)} + a_k^{(2)}M_k^{(2)}), \quad (5.4)$$

$$M_k^{(1)} = \frac{1}{y_1 Q}\left[\varepsilon^{-(\nu+2)} l^- \left(1 - \frac{1}{\nu}\right)\left(\frac{y}{y_2}\right)^{\nu-2} - l^+\left(1 + \frac{1}{\nu}\right)\left(\frac{y_1}{y}\right)^{\nu+2}\right]\cos(\nu\varphi), \quad (5.5)$$

$$M_k^{(2)} = \frac{1}{y_1 Q \varepsilon}\left[q^+\left(1 - \frac{1}{\nu}\right)\left(\frac{y}{y_2}\right)^{\nu-2} - \varepsilon^{-\nu+2} q^-\left(1 + \frac{1}{\nu}\right)\left(\frac{y_2}{y}\right)^{\nu+2}\right]\cos(\nu\varphi), \quad (5.6)$$

$$q^\pm = \frac{g_1 y_1}{\nu} \pm d_1, \quad l^\pm = \frac{g_2 \varepsilon y_1}{\nu} \pm d_2, \quad Q = p^+ l^+ - p^- l^- \varepsilon^{-2\nu},$$

$$k_0 = 1 + E(\sqrt{\lambda_1/\lambda_2}),$$

in which E(...) denotes the integer part of (...). We have $\nu > 1$ for all $k > k_0$. Then it is readily shown that

$$|M_k^{(1)}| \leqslant c', \quad |M_k^{(2)}| \leqslant c'',$$

$$c' = \frac{3(d_2 + g_2 \varepsilon y_1)\varepsilon^2}{d_1 d_2 y_1(\varepsilon^2 - 1)}, \quad c'' = \frac{3(d_1 + g_1 y_1)\varepsilon^2}{d_1 d_2 y_1(\varepsilon^2 - 1)}.$$

Then the majorant series of (5.4) will be

$$\sum_{k=k_0}^{\infty} \nu\left(|a_k^{(1)}|c' + |a_k^{(2)}|c''\right). \quad (5.7)$$

Now $a_k^{(1)}$ and $a_k^{(2)}$ are the Fourier coefficients of the given boundary functions F_1 and F_2. If these periodic boundary functions are continuous, together with their derivatives up to order s (s > 2), it is readily seen that the majorants of (5.7) converge, and hence that the series of (5.4) converges uniformly.

The most unfavorable case as regards uniform convergence in (5.4) occurs when F_1 and F_2 are not continuous, although they satisfy the Dirichlet conditions. Then we can only say that the $a_k^{(1)}$ and $a_k^{(2)}$ for large k will be infinitely small quantities of order not less than 1/k; but this is insufficient for convergence of the majorants of (5.7). If this approximation is impossible, we are forced to accept uniform convergence of (5.4) only within the region $y_2 > y > y_1$.

Consider the region $\alpha_2 \geq y \geq \alpha_1$, where $\alpha_2 < y_2$, $\alpha_1 > y_1$.

Let $\nu^* > 1$ denote the value of ν that for all $\nu > \nu^*$ provides simultaneously

$$(\alpha_2/y_2)^{\nu-2} \leqslant \nu^{-s}, \quad (y_1/\alpha_1)^{\nu+2} \leqslant \nu^{-s} \quad (s > 1).$$

Then we may show that

$$|M_k^{(1)}| \leqslant c'\nu^{-s}, \quad |M_k^{(2)}| \leqslant c''\nu^{-s} \quad (k > k_0).$$

This is sufficient for convergence of the majorants of (5.7), and hence for uniform convergence of (5.4), within the region $y_2 > y > y_1$.

If in (5.4) we use the following in place of (5.5) and (5.6),

$$M_k^{(1)} = -\frac{y_1}{Q}\left[\varepsilon^{-\nu} l^-\left(\frac{y}{y_2}\right)^{\nu} - l^+\left(\frac{y_1}{y}\right)^{\nu}\right]\cos(\nu\varphi),$$

$$M_k^{(2)} = -\frac{y_2}{Q}\left[q^+\left(\frac{y}{y_2}\right)^{\nu} - \varepsilon^{-\nu} q^-\left(\frac{y_1}{y}\right)^{\nu}\right]\cos(\nu\varphi),$$

we get series S_2 of (5.1), and the proof of uniform convergence for this in no way differs from the above proof for S_1.

Consider the double series S_3 of (5.1). We set

$$M_k^{(1)} = \frac{2y_1\xi_{kn}}{\Delta}\left\{\left[-d_2\varepsilon N_{\nu+1}(\varepsilon\xi_{kn}) + \frac{\nu l^+}{\xi_{kn}} N_\nu(\varepsilon\xi_{kn})\right] \times \left[\left(\frac{\nu^2 - \nu}{\xi_{kn}^2}\frac{y_1^2}{y^2} - 1\right) J_\nu(\xi_{kn} y/y_1) + \frac{y_1}{y\xi_{kn}} J_{\nu+1}(\xi_{kn} y/y_1)\right] + \left[d_2\varepsilon J_{\nu+1}(\varepsilon\xi_{kn}) - \frac{\nu l^+}{\xi_{kn}} J_\nu(\varepsilon\xi_{kn})\right] \times \left[\left(\frac{\nu^2 - \nu}{\xi_{kn}^2}\frac{y_1^2}{y^2} - 1\right) N_\nu(\xi_{kn} y/y_1) + \frac{y_1}{y\xi_{kn}} N_{\nu+1}(\xi_{kn} y/y_1)\right]\right\},$$

$$M_k^{(2)} = \frac{2\varepsilon y_1\xi_{kn}}{\Delta}\left\{\left[d_1 N_{\nu+1}(\xi_{kn}) + \frac{\nu q^-}{\xi_{kn}} N_\nu(\xi_{kn})\right] \times \left[\left(\frac{\nu^2 - \nu}{\xi_{kn}^2}\cdot\frac{y_1^2}{y^2} - 1\right) J_\nu(\xi_{kn} y/y_1) + \frac{y_1}{y\xi_{kn}} J_{\nu+1}(\xi_{kn} y/y_1)\right] + \left[-d_1 J_{\nu+1}(\xi_{kn}) - \frac{\nu q^-}{\xi_{kn}} J_\nu(\xi_{kn})\right] \times \left[\left(\frac{\nu^2 - \nu}{\xi_{kn}^2}\frac{y_1^2}{y^2} - 1\right) N_\nu(\xi_{kn} y/y_1) + \frac{y_1}{y\xi_{kn}} N_{\nu+1}(\xi_{kn} y/y_1)\right]\right\}$$

$$(\Delta = \Delta_{kn}'/\xi_{kn}^2 \neq 0).$$

The result is

$$\frac{\partial^2 v_{kn}}{\partial y^2} = (a_k^{(1)} M_k^{(1)} + a_k^{(2)} M_k^{(2)}) \exp\frac{-\xi_{kn}^2\tau}{m^2 y_1^2}\cos(\nu\varphi).$$

As ν increases without limit, the roots ξ_{kn} of $\Delta_k(\xi) = 0$ increase at least as $c_1\nu$, with $c_1 > 0$ [5]; also, using the limiting relations

$$\lim_{x\to\infty} |\sqrt{x} J_\nu(x)| < {}^1/_2\pi > \lim_{x\to\infty} |\sqrt{x} N_\nu(x)|$$

it may be shown that

$$\lim_{k\to\infty} |M_k^{(1)}| \leqslant \frac{8d_2 y_1}{\pi\sqrt{\varepsilon c_1^3}\Delta}(1 + \varepsilon c_1)(1 + c_1^2),$$

$$\lim_{k\to\infty} |M_k^{(2)}| \leqslant \frac{8d_1\varepsilon y_1}{\pi c_1^2\Delta}(1 + c_1)(1 + c_1^2),$$

i.e., as k increases without limit, the coefficients $a_k^{(1)} M_k^{(1)} + a_k^{(2)} M_k^{(2)}$ of S_3 are not merely bounded but also tend to zero at least as 1/k. Then it is obvious that it is sufficient to demonstrate convergence of the majorants

$$\sum_{k=k'}^{\infty}\sum_{n=1}^{\infty} \exp\frac{-c^2(k+n)^2\tau}{m^2 y_1^2}$$

in order to demonstrate uniform convergence of S_3.

Here we should note that the roots ξ_{kn} of $\Delta_k(\xi) = 0$ increase at least as $c(k + n)$, with $c > 0$ [5], as k or n increases without limit.

Uniform convergence of the double series S_4 may be demonstrated similarly.

The above proofs of uniform convergence for the series of (5.1) apply also to the series S_i^* derived from S_i by replacing f_k, C_{kn}, D_{kn}, $\cos(\nu\varphi)$, respectively, by

$$f_k^*, \; C_{kn}^*, \; D_{kn}^*, \quad \sin(\nu\varphi).$$

This completes the proof of uniform convergence for the series in (4.5).

After this paper had been accepted, Cinelli's paper [6] came to my notice, in which the finite Hankel transform with respect to the spatial coordinates is used to consider a problem analogous to the one solved here via the Laplace integral transform with respect to the time coordinate.

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